Current Mathematics Appears to Be Inconsistent

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Abstract

We show that some mathematical results and their negations are both deducible. The derived contradictions indicate the inconsistency of current mathematics.

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According to Gödel's second incompleteness theorem, if mathematics is consistent, the consistency cannot be proved by mathematical means [2]. However, to show the inconsistency of mathematics, it is sufficient to deduce both a mathematical statement and its negation. This paper gives some examples of such statements. We first consider an example in a geometric setting.

Let line l be the x-axis on the Euclidean plane. Let $X_n, n = 1, 2, \cdots$ be points on l with coordinates $(x_n, 0)$, i.e., $X_n = (x_n, 0)$, where $x_n, n = 1, 2, \cdots$ form a strictly decreasing sequence of positive numbers, converging to a unique limit a. Denote by A the point $(a, 0)$ on l, and B the point (a, b) on the plane, where b is an arbitrarily given positive number (Figure 1).

Statement 1 *There is a line segment AA' on l from point $A = (a, 0)$ to point $A' = (a', 0)$, where $a' > a$, such that there is not any point $X_n = (x_n, 0)$ on AA'.*

Proof: For each point $X_n = (x_n, 0)$ with $n > 1$, there is a line segment X_nB from X_n to B. Since $x_n > a$ for any x_n, no X_nB is perpendicular to l. Let X_nBX_1 represent the set of the points that form the triangle with sides X_nB, BX_1, and X_1X_n. Denote by Δ the union of all such triangles X_nBX_1. Let ABX_1 represent the set of the points that form the triangle with sides AB, BX_1, and X_1A. The line passing through both A and B is perpendicular to l. No X_nB coincides with AB. So Δ is a proper subset of ABX_1 (Figure 1).

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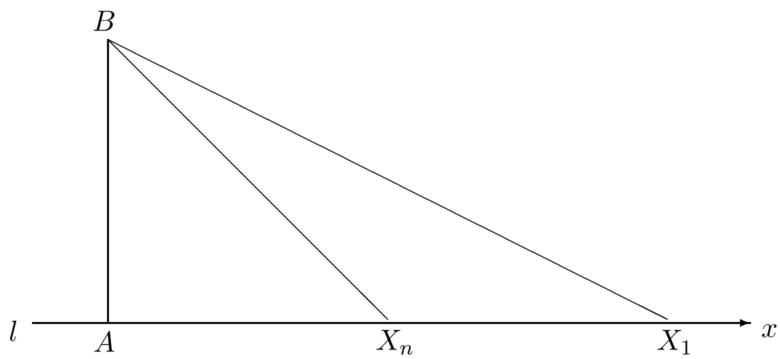

Figure 1. The geometric setting.

Denote by $m(ABX_1 \setminus \Delta)$ the value of the Lebesgue measure (on the plane) of $ABX_1 \setminus \Delta$. There are two conceivable cases. If $m(ABX_1 \setminus \Delta) > 0$, then there is a segment AA' on l from A to a point $A' = (a', 0) \in ABX_1 \setminus \Delta$ with $a < a' < x_n$ for each x_n. There is not any point X_n on AA'.

The complement (relative to l) of the set $\{X_n : n \geq 1\} \cup \{A\}$ is a union of a collection $\mathcal{I}$ of disjoint open intervals on l. Though the lengths of the finite open intervals in $\mathcal{I}$ converge to 0, no $I \in \mathcal{I}$ is of length zero. If $m(ABX_1 \setminus \Delta) = 0$, then there is an open interval $I_a \in \mathcal{I}$ with its left endpoint coinciding with A (otherwise $m(ABX_1 \setminus \Delta) > 0$). Let $A' = (a', 0)$ be a point in I_a. The segment AA' does not contain any point X_n. **Q.E.D.**

Statement 2 *There is not a line segment AA' on l from point $A = (a, 0)$ to point $A' = (a', 0)$, where $a' > a$, such that there is not any point $X_n = (x_n, 0)$ on AA'.*

Proof: The point A is the accumulation point of the set $\{X_n : n \geq 1\}$ on l. The definition of accumulation point rules out immediately the existence of such AA'. **Q.E.D.**

The Statements 1 and 2 can also be formulated and proved in an analytical setting. Consider a set consisting of terms $x_n, n = 1, 2, \cdots$ of a strictly decreasing sequence of positive numbers with limit a. The set $\{x_n : n \geq 1\} \cup \{a\}$ is a closed set. The complement of $\{x_n : n \geq 1\} \cup \{a\}$ is a union of a collection $\mathcal{I}$ of disjoint open intervals.

Statement 3 *There is an open interval $I_a \in \mathcal{I}$, such that the left endpoint of I_a coincides with a.*

Proof: Denote by $\mathcal{J}$ the sub-collection of all finite open intervals in $\mathcal{I}$. Since

$$\inf\{|a - x| : x \in \cup_{I \in \mathcal{J}}\} = \inf\{|a - x| : x \in I,\ I \in \mathcal{J}\} = 0$$

there is an $I_a \in \mathcal{J}$ with

$$\inf\{|a - x| : x \in I_a\} = 0.$$

The lower endpoint of I_a is a. Actually, such $I_a \in \mathcal{J}$ with $\inf I_a = a$ is a consequence of the result below.

$$\inf \bigcup_{I\in\mathcal{J}} I = \inf\{x : x \in I,\ I \in \mathcal{J}\} = a.$$

Q.E.D.

The following statement is an immediate consequence of the definition of accumulation point.

Statement 4 *There is not an open interval $I_a \in \mathcal{I}$, such that the left endpoint of I_a coincides with a.*

A variant of Statement 3 is as follows.

Statement 5 *There is an open interval $I_a \in \mathcal{J}$ with a minimum length.*

We first emphasize the difference between the limit of a convergent sequence (with different terms) and any term of the sequence. Though the lengths of the open intervals in $\mathcal{J}$ converge to 0, no $I \in \mathcal{J}$ is of length zero. Replacing the positive length of any $I \in \mathcal{J}$ with the limit 0 is strictly prohibited in current mathematics.

Proof: Consider a thought experiment. For each open interval $I \in \mathcal{J}$, there is a timer with a pre-set time period equal to the length of I. All timers start at the same instant, and also stop simultaneously once the time period of any of the timers expires. Let τ be the time elapsed between the starting and stopping instants. Since $\tau = 0$ implies an open interval of a zero length, which contradicts the Archimedean axiom, we have $\tau > 0$. There is then an open interval $I_a \in \mathcal{J}$ with a minimum length τ.
Q.E.D.

Statement 6 *There is not an open interval $I_a \in \mathcal{J}$ with a minimum length.*

Proof: Let $|I|$ represent the length of an interval I. If there is an open interval $I_a \in \mathcal{J}$ with a minimum length τ, then

$$\sum_{I\in\mathcal{J}} |I| \geq \sum_{I\in\mathcal{J}} \tau.$$

The right side is infinite. But

$$\sum_{I\in\mathcal{J}} |I| = x_1 - a \tag{1}$$

is finite. **Q.E.D.**

Current mathematics provides yet another way to prove Statement 6 by an assertion: A countable intersection of infinitely many nested open intervals with a common endpoint is an empty set. This assertion is also an immediate consequence of the definition of accumulation point or limit.

Similar to the proofs of Statements 5 and 6, the following two statements are both deducible. Denote by $\mathcal{K}$ the collection of the open intervals such that $\cup_{I\in\mathcal{K}} I$ is the complement of the Cantor set relative to the closed unit interval $[0, 1]$.

Statement 7 *There is an open interval $I \in \mathcal{K}$ with a minimum length.*

Statement 8 *There is not an open interval $I \in \mathcal{K}$ with a minimum length.*

Some statements above follow immediately from definitions or axioms adopted in current mathematics. The derivations of the other statements rely on some intermediate results, which can be found in standard textbooks. All the statements are logical consequences of current mathematics, and hence are mathematically true. It is meaningless to use one of the statements (or the definitions and axioms behind the statement) to argue against the other. The contradictions between the statements should only be interpreted as the indication of the inconsistency of current mathematics.

There might be some psychological difficulties in conceiving the inconsistency. The inconsistency involves the notion of accumulation point or limit, which in turn relies on the notion of infinite. There are two different conceptions of infinite in mathematics: potential infinite versus actual infinite. A difficulty might originate from potential infinite.

For a convergent sequence of different numbers, the notion of limit is formulated by the (N, ϵ) argument ("for each positive real number ϵ, there is a positive integer N ..."). Different conceptions of infinite lead to different interpretations of the formulation. The interpretation of limit based on potential infinite goes like this. For each specified positive number ϵ, a positive integer N can be found with the required property, and this process of "finding an N for each specified ϵ" will never end. With such interpretation of limit, one might deny the existence of the collection $\mathcal{I}$ (or $\mathcal{K}$) of the open intervals in the first place. But in the eyes of anyone who is trained in the set-theoretic tradition, such denial might be absurd, since the existence of $\mathcal{I}$ (or $\mathcal{K}$) seems to be an obvious consequence of the existence of the set of all real numbers.

On the other hand, from the viewpoint of actual infinite, the (N, ϵ) formulation of limit contains, as remarked by Abraham Robinson, "a clear reference to a well-defined infinite totality, i.e., the totality of positive real numbers" [3]. However, even for those who agree with the viewpoint of actual infinite, there might be yet a difficulty in conceiving the inconsistency.

For example, consider (1). The right side of (1) is the limit of a sequence of partial sums. The interpretation of limit based on the notion of actual infinite, in this example, treats $\mathcal{J}$ as a completed totality, and the limit as the total length of the open intervals in $\mathcal{J}$. So the length of each open interval in $\mathcal{J}$ is a positive addendum of an infinite sum. The limit is a result of adding together all the addenda without leaving any out. However, is there a last addendum added into the total? In other words, is there an open interval in $\mathcal{J}$ with a minimum length?

By Statement 5, there is an open interval in $\mathcal{J}$ with a minimum length. As shown in the proof of Statement 5, the non-existence of such open interval implies an open interval of length zero, which contradicts the Archimedean axiom. But the existence of an open interval in $\mathcal{J}$ with a minimum length implies a maximum natural number, which contradicts Peano's axioms. The basis of natural numbers and any reasoning involving natural numbers is a set of assumptions, called Peano's axioms. The Archimedean axiom involves natural numbers, and hence is also a consequence of Peano's axioms. The inconsistency of current mathematics revealed in this paper suggests a fundamental flaw in the formulation of the notion of natural numbers.

Many mathematicians share a belief: All mathematical statements should be reducible ultimately to statements about natural numbers [1]. Perhaps it is this belief that makes the inconsistency of current mathematics difficult to detect.

References

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